Commutation in Vector Spaces over Division Rings with a Conjugation*

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ABSTRACT

We analyze some commutation properties of the sets of mappings of a vector space X over a division ring K with a conjugation j which are relevant when studying symmetries in quantum mechanics and in elementary-particle physics. The first part of the paper is devoted to the "linear-antilinear centralizer" $\mathfrak{A}^c$, i.e. to the group of the linear and antilinear (j-semilinear) invertible mappings which commute with a given set $\mathfrak{A}$ of mappings of X. Some nontrivial results which connect properties of $\mathfrak{A}$ with properties of $\mathfrak{A}^c$ are obtained, and a classification of the sets of mappings of X is found by means of purely algebraic techniques. This classification is more detailed than that usually adopted by physicists. The second part of the paper is devoted to the Λ-linear commutant $\mathfrak{A}'^{\lambda}$, i.e. to the set of mappings of X which commute with $\mathfrak{A}$ and which are linear with respect to the j-invariant subring Λ of K. We investigate the structure of $\mathfrak{A}'^{\lambda}$ in connection with the structure and some of the properties of $\mathfrak{A}$. In the third part, we show how the results obtained in the preceding sections simplify when the division ring K is of type II (according to a classification introduced in an earlier work). Finally, we illustrate with simple examples in one- and two-dimensional vector spaces all the cases which can occur.

1. INTRODUCTION

The present work is part of some research which began by seeking to generalize some linear-algebra propositions which are important for a number of reasons in theoretical physics and which have been stated in literature only for vector spaces over the complex field C. In an earlier work on this subject [1] special attention was given to the classification of complex group representations introduced by Frobenius and Shur in 1906 [2] and reported in more modern form in several books on group theory [3, Chapter 8; Problems, p. 378], especially those written for physicists [4, Sec. 24, p. 285;

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5, Chapter 5, §5; 6, Sec. 23]. Hence, we were led to identify the relevant structure as a vector space X over a division ring K (here and in the sequel the term "division ring" has the meaning specified in Ref. [7]; in some previous works [1, 8] we used instead the term "field" with this meaning) with a nonidentical involutory automorphism $j: \alpha \to \bar{\alpha}$ (conjugation), and a subsequent work [8] was partially devoted to a deeper and systematical investigation of this structure. Indeed, we recall that the pair consisting of a vector space X over a division ring with a conjugation, together with a set $\mathfrak{A}$ of linear and antilinear (i.e. j-semilinear) mappings, may be considered as a projective geometry with a given set of operators. Thus, the elements of the linear-antilinear centralizer $\mathfrak{A}^c$ (i.e. the group of the linear and antilinear invertible mappings of X that commute with $\mathfrak{A}$) describe automorphisms of this structure; moreover, when finite-dimensional vector spaces over the complex field are considered, they represent (provided some continuity requirement is added or the rational complex field is considered) all the possible automorphisms. This point is relevant in quantum physics, where the symmetries of a physical system are described by automorphisms of a complex vector space (endowed with an orthocomplementation) with some continuity requirements and some complications in the infinite-dimensional case that we do not report here; indeed, the elements of the linear-antilinear centralizer (preserving the orthocomplementation) eventually describe further symmetries of the system ("internal" symmetries) [8, §1, p. 133; note (4)].

The main purpose of this paper is to develop the connection, stated in Ref. [1], between some properties of a given set $\mathfrak{A}$ of mappings of X and the properties of $\mathfrak{A}^c$, and to extend it to the Λ-linear commutant $\mathfrak{A}'^{\lambda}$ of $\mathfrak{A}$ (i.e. the set of all the mappings of X that commute with $\mathfrak{A}$ and that are linear with respect to the j-invariant subring Λ of K), using the results obtained in Ref. [8].

Indeed, a set of conditions regarding $\mathfrak{A}$ (and $\mathfrak{A}^c$) were stated in Ref. [1] which are equivalent to the requirement that $\mathfrak{A}^c$ be isomorphic to a semidirect product $\mathfrak{A}^{cl} \circledS G_2$ of its linear part $\mathfrak{A}^{cl}$ and the two-element group G_2 (in particular, when any $A \in \mathfrak{A}$ happens to be linear or antilinear, an equivalent condition on this set states that a basis $\mathfrak{S}$ exists such that the matrix elements of the representations of $\mathfrak{A}$ in $\mathfrak{S}$ belong to Λ) [1, Sec. 3, Sec. 4].

In the first part of this paper (Sec. 2), we first observe that $\mathfrak{A}^c$ can easily be constructed whenever its linear part $\mathfrak{A}^{cl}$ and an arbitrary antilinear mapping of $\mathfrak{A}^c$ are known (Proposition 1). Then, we show that the mapping of $\mathfrak{A}^{cl}$ onto itself associated with the nonidentical element of G_2 when $\mathfrak{A}^c$ is isomorphic to the aforesaid semidirect product [more generally, any mapping $\theta(A)$: $L \in \mathfrak{A}^{cl} \to ALA^{-1} \in \mathfrak{A}^{cl}$, with A belonging to the antilinear part $\mathfrak{A}^{ca}$ of

TABLE 1[a]

$\mathcal{U}^{ca}$ is nonvoid (hence, $\mathcal{U}^{c}=\mathcal{U}^{cl}\cup A\mathcal{U}^{cl}=A\mathcal{U}^{ca}\cup\mathcal{U}^{ca}$ for any $A\in\mathcal{U}^{ca}$)	The equivalent statements of Proposition 2 hold (in particular a mapping $J\in\mathcal{U}^{ca}$ exists such that $J^2=E$)	The equivalent statements of Proposition 3 hold [in particular, $(\mathcal{U}\cup\mathcal{U}^{c})^{ca}\neq\varnothing$]	The equivalent statements of Proposition 4 hold [in particular, a mapping $J\in(\mathcal{U}\cup\mathcal{U}^{c})^{ca}$ exists such that $J^2=E$]	For any $J\in\mathcal{J}=\{J\in(\mathcal{U}\cup\mathcal{U}^{c})^{ca}: J^2=E\}$, $\mathcal{U}^{c}=\mathcal{U}^{cl}\times\{E,J\}$. For any $A\in\mathcal{U}^{ca}$, $\theta(A)$ is inner. (Hence, $\mathcal{U}^{c}$ is commutative if and only if $\mathcal{U}^{cl}$ is also commutative.)
			The equivalent statements of Proposition 4 do not hold [in particular, for any $A\in(\mathcal{U}\cup\mathcal{U}^{c})^{ca}$, $A^2\neq E$]	For any $J\in\mathcal{J}=\{J\in\mathcal{U}^{ca}: J^2=E\}$, $\mathcal{U}^{c}=\mathcal{U}^{cl}\circledS_{\theta}\{E,J\}$ with $\theta(J)\neq 1$. For any $A\in\mathcal{U}^{ca}$, $\theta(A)$ is inner (and depends on A). (Hence, $\mathcal{U}^{c}$ is noncommutative.)
		The equivalent statements of Proposition 3 do not hold [in particular, $(\mathcal{U}\cup\mathcal{U}^{c})^{ca}=\varnothing$]		For any $J\in\mathcal{J}=\{J\in\mathcal{U}^{ca}: J^2=E\}$, $\mathcal{U}^{c}=\mathcal{U}^{cl}\circledS_{\theta}\{E,J\}$. For any $A\in\mathcal{U}^{ca}$, $\theta(A)$ is not inner. (Hence, $\mathcal{U}^{c}$ is noncommutative.)
	The equivalent statements of Proposition 2 do not hold (in particular, for any $A\in\mathcal{U}^{ca}$, $A^2\neq E$)	The equivalent statements of Proposition 3 hold [in particular, $(\mathcal{U}\cup\mathcal{U}^{c})^{ca}\neq\varnothing$]		$\mathcal{U}^{cl}\neq\mathcal{U}^{c}\nsim\mathcal{U}^{cl}\circledS G_2$. For any $A\in\mathcal{U}^{ca}$, $\theta(A)$ is inner. ($\mathcal{U}^{c}$ is commutative if and only if $\mathcal{U}^{cl}$ is also commutative.)
		The equivalent statements of Proposition 3 do not hold [in particular, $(\mathcal{U}\cup\mathcal{U}^{c})^{ca}=\varnothing$]		$\mathcal{U}^{cl}\neq\mathcal{U}^{c}\nsim\mathcal{U}^{cl}\circledS G_2$. For any $A\in\mathcal{U}^{ca}$, $\theta(A)$ is not inner. (Hence, $\mathcal{U}^{c}$ is noncommutative.)
$\mathcal{U}^{ca}$ is void				$\mathcal{U}^{c}=\mathcal{U}^{cl}$

[a] 1 is the identity mapping of $\mathcal{U}^{cl}$; ⓢ means semidirect product; $\theta(A)$ is the mapping of $\mathcal{U}^{cl}$ into itself such that $L \xrightarrow{\theta} ALA^{-1}$; G_2 is the abstract two-element group; $\sim$ means group isomorphism; E is the identity mapping of X.

$\mathfrak{A}^c$] is inner if and only if the center $(\mathfrak{A} \cup \mathfrak{A}^c)^c$ of $\mathfrak{A}^c$ contains some antilinear mappings (Proposition 3); furthermore we state (Proposition 4) a set of conditions concerning $\mathfrak{A} \cup \mathfrak{A}^c$ which are equivalent to the requirement that $\mathfrak{A}^c$ be isomorphic to the direct product $\mathfrak{A}^{cl} \times G_2$ (in particular, when any $A \in \mathfrak{A}$ happens to be linear or antilinear, this isomorphism occurs if and only if a basis $\mathfrak{S}$ exists such that the matrix elements of the representation of $\mathfrak{A} \cup \mathfrak{A}^c$ in $\mathfrak{S}$ belong to Λ). Table 1 summarizes all these results and gives a refinement of the classification of the representations obtained in a previous work [1, Sec. 3, Theorem 4] (in particular, the case $\mathfrak{A}^c \sim \mathfrak{A}^{cl} \circledS G_2$ splits here into three subcases, and the case $\mathfrak{A}^c \nsim \mathfrak{A}^{cl} \circledS G_2$ splits into two subcases).

In the second part of the paper (Sec. 3), introducing suitable further assumptions about $\mathfrak{A}$ (chiefly irreducibility), our main result is that $\mathfrak{A}^c$ is isomorphic to a semidirect product of $\mathfrak{A}^{cl}$ and G_2 if and only if the ring $\mathfrak{A}'^{\Lambda}$ contains divisors of zero (Proposition 7).

Finally, we show in the last sections how the results of the preceding sections simplify when the division ring K is suitably particularized (Sec. 4) and illustrate with simple examples all the cases which can occur (Sec. 5).

We will prove in a forthcoming paper that whenever $\mathfrak{A}$ is an irreducible group of linear and antilinear mappings (some weaker conditions for $\mathfrak{A}$ are actually sufficient), X finite-dimensional, and K an algebraically closed field, $\mathfrak{A}^{cl}$ can be explicitly constructed; then, the results obtained here can be used to exhibit $\mathfrak{A}^c$ whenever any $A \in \mathfrak{A}^{ca}$ is known, and to clarify its group or ring structure further. Whenever $K = C$ (complex field), we also obtain many results which in the literature have been achieved in quite a different framework [9] (see also [10]).

2. THE LINEAR-ANTILINEAR CENTRALIZER

Here we collect some definitions that have already been given in other papers on the subject and that will be used throughout the present work.

DEFINITION 1 [8, §2, Definition 1]. We call any division ring K endowed with a nonidentical involutory automorphism $j: \alpha \to \bar{\alpha}$ a *division ring with a conjugation*. We call the subdivision ring Λ of K which consists of the self-conjugated elements of K the *j-invariant subring* of K.

DEFINITION 2 [1, Sec. 2, Definition 2]. Let X be a vector space over a division ring K with a conjugation j. Then we call any additive mapping of X which is semilinear with respect to j an *antilinear mapping*.

Let $\mathfrak{S}$ be a basis in X. We call the antilinear mapping that leaves the elements of $\mathfrak{S}$ invariant *conjugation in X associated with the basis* $\mathfrak{S}$, and denote it by $J_{\mathfrak{S}}$. (Here and in the sequel we denote by E the identity automorphism of X; then $J_{\mathfrak{S}}^2 = E$ and, conversely, every antilinear involutory mapping $A \in \mathfrak{U}^{ca}$ is a conjugation in X [1, Sec. 3, Lemma 1]. The index $\mathfrak{S}$ will be omitted whenever the basis need not be mentioned.)

For any mapping A of X and any basis $\mathfrak{S}$, we define the *conjugate mapping* $\bar{A}_{\mathfrak{S}} = J_{\mathfrak{S}} A J_{\mathfrak{S}}^{-1}$ of A with respect to the basis $\mathfrak{S}$.

Definition 3 [1, Sec. 3, Definition 3; 8, §3, Definition 2]. Let X be a vector space over a division ring K, and let $\mathfrak{U}$ be any set of mappings of X. We denote the subset of all the linear mappings of $\mathfrak{U}$ by $\mathfrak{U}^l$. Moreover, let K be a division ring with a conjugation. We denote the subset of all the antilinear (Λ-linear) mappings of $\mathfrak{U}$ by $\mathfrak{U}^a$ ($\mathfrak{U}^\lambda$), and we put $\mathfrak{U}^h = \mathfrak{U}^l \cup \mathfrak{U}^a$. We call the multiplicative group of all the linear and antilinear invertible mappings of X which commute with all the mappings of $\mathfrak{U}$ the *linear-antilinear centralizer* of $\mathfrak{U}$ and denote it by $\mathfrak{U}^c$. Hence, $\mathfrak{U}^{cl}$ (the *linear centralizer* of $\mathfrak{U}$) is the subgroup of all the linear mappings of $\mathfrak{U}^c$, and $\mathfrak{U}^{ca}$ (the *antilinear centralizer* of $\mathfrak{U}$) the subset of all the antilinear mappings of $\mathfrak{U}^c$.

Proposition 1. *Let X be a vector space over a division ring K with a conjugation j. With reference to Definitions* 1–3, *let $\mathfrak{U}$ be a set of mappings of X; then for any $A \in \mathfrak{U}^{ca}$ we have $\mathfrak{U}^{ca} = \mathfrak{U}^{cl}A = A\mathfrak{U}^{cl}$, and whenever $\mathfrak{U}^{ca}$ is nonvoid, $\mathfrak{U}^{cl} = \mathfrak{U}^{ca}A = A\mathfrak{U}^{ca}$.*

Moreover, $(\mathfrak{U} \cup \mathfrak{U}^c)^{ca} = (\mathfrak{U} \cup \mathfrak{U}^{cl})^{ca}$ (hence, whenever $(\mathfrak{U} \cup \mathfrak{U}^c)^{ca}$ is nonvoid, $(\mathfrak{U} \cup \mathfrak{U}^c)^c$, which is the center of $\mathfrak{U}^c$, coincides with $(\mathfrak{U} \cup \mathfrak{U}^{cl})^c$).

Proof. Let $\mathfrak{V}$ be any multiplicative semigroup of mappings of X, and let an invertible mapping A exist such that A and A^{-1} belong to $\mathfrak{V}^a$; then $\mathfrak{V}^a = \mathfrak{V}^l A = A\mathfrak{V}^l$ and $\mathfrak{V}^l = \mathfrak{V}^a A = A\mathfrak{V}^a$. Indeed, for any $B \in \mathfrak{V}^a$, $B = (BA^{-1})A$; since $\mathfrak{V}$ is a semigroup and $A^{-1} \in \mathfrak{V}^a$, we obtain $BA^{-1} \in \mathfrak{V}^l$, so that $\mathfrak{V}^a \subseteq \mathfrak{V}^l A$. Conversely, $\mathfrak{V}^l A \subseteq \mathfrak{V}^a$, where $\mathfrak{V}$ is a semigroup. Thus, $\mathfrak{V}^a = \mathfrak{V}^l A$ and, analogously, $\mathfrak{V}^a = A\mathfrak{V}^l$. Hence, $\mathfrak{V}^l = \mathfrak{V}^a A = A\mathfrak{V}^a$.

For any set $\mathfrak{U}$ of mappings of X such that a mapping $A \in \mathfrak{U}^{ca}$ exists, let us consider $\mathfrak{U}^c$; this obviously satisfies the conditions assumed for $\mathfrak{V}$ in the above observation, so that $\mathfrak{U}^{ca} = \mathfrak{U}^{cl}A = A\mathfrak{U}^{cl}$ and $\mathfrak{U}^{cl} = \mathfrak{U}^{ca}A = A\mathfrak{U}^{ca}$.

Now, observe that $(\mathfrak{U} \cup \mathfrak{U}^c)^c \subseteq (\mathfrak{U} \cup \mathfrak{U}^{cl})^c$; furthermore, let $(\mathfrak{U} \cup \mathfrak{U}^{cl})^{ca}$ be void; then $(\mathfrak{U} \cup \mathfrak{U}^c)^{ca}$ is void, and hence $(\mathfrak{U} \cup \mathfrak{U}^c)^{ca} = (\mathfrak{U} \cup \mathfrak{U}^{cl})^{ca}$. Let an $A \in (\mathfrak{U} \cup \mathfrak{U}^{cl})^{ca}$ exist; then $(\mathfrak{U} \cup \mathfrak{U}^{cl})^c = (\mathfrak{U} \cup \mathfrak{U}^{cl})^{cl} \cup (\mathfrak{U} \cup \mathfrak{U}^{cl})^{cl}A$ because of the statement proved above, and any element of this last set commutes with $\mathfrak{U}^c = \mathfrak{U}^{cl} \cup \mathfrak{U}^{cl}A$ (as can easily be checked), so that $(\mathfrak{U} \cup \mathfrak{U}^{cl})^c \subseteq (\mathfrak{U} \cup \mathfrak{U}^c)^c$; hence $(\mathfrak{U} \cup \mathfrak{U}^{cl})^{ca} = (\mathfrak{U} \cup \mathfrak{U}^c)^{ca}$ again. [The statement in parentheses directly follows from our proof; alternatively, it can be obtained from our last equation by multiplying it by $A \in (\mathfrak{A} \cup \mathfrak{A}^{cl})^{ca}$ and observing that it then gives $(\mathfrak{A} \cup \mathfrak{A}^{cl})^{cl} = (\mathfrak{A} \cup \mathfrak{A}^{c})^{cl}$.] ■

PROPOSITION 2. *Let X be a vector space over a division ring K with a conjugation j. With reference to Definitions 1–3, let $\mathfrak{A}$ be a set of mappings of X; then the following statements are equivalent:*

(i) *a basis $\mathfrak{E}$ exists in which, for any $A \in \mathfrak{A}$, $\bar{A}_{\mathfrak{E}} = A$,*[1]

(ii) *a basis $\mathfrak{E}$ exists in which, for any $A \in \mathfrak{A}$, $\bar{A}_{\mathfrak{E}} = SAS^{-1}$, where S is a linear mapping of X such that $S\bar{S}_{\mathfrak{E}} = E$,*

(iii) *in any basis $\mathfrak{E}$, for any $A \in \mathfrak{A}$, we have $\bar{A}_{\mathfrak{E}} = SAS^{-1}$, with S a linear mapping of X such that $S\bar{S}_{\mathfrak{E}} = E$,*

(iv) *an involutory antilinear mapping exists which belongs to $\mathfrak{A}^{c}$,*

(v) *a mapping $A \in \mathfrak{A}^{ca}$ exists such that the set $\{L \in \mathfrak{A}^{cl} : L^2 = A^2$ and $LA = AL\}$ is nonvoid,*

(vi) *the linear-antilinear centralizer $\mathfrak{A}^{c}$ of $\mathfrak{A}$ is isomorphic to a semidirect product of the linear centralizer $\mathfrak{A}^{cl}$ and the two-element group G_2.*

Furthermore, let the equivalent statements listed above hold. Denote by $\tilde{G}_2$ a two-element group whose elements are E and any one of the antilinear involutory mappings J whose existence is assured by (iv), *and by θ the mapping of $\tilde{G}_2$ into the group of automorphism of $\mathfrak{A}^{cl}$ such that $\theta(E) = 1$ (identity mapping of $\mathfrak{A}^{cl}$) and $\theta(J): L \in \mathfrak{A}^{cl} \to JLJ^{-1} \in \mathfrak{A}^{cl}$.*[2] *Then $\mathfrak{A}^{c}$ is canonically identical to the semidirect product $\mathfrak{A}^{cl} \circledS_{\theta} \tilde{G}_2$ (through the identifications $(L, E) \to L$ and $(L, J) \to LJ$).*

Proof. The equivalence of statements (i), (ii), (iii), (iv) and (vi) and the final statement of the proposition have been proved in a previous work [1, Sec. 3, Theorem 3]. Therefore, we need only show that (v) is also equivalent to the other statements. To this end, let us observe that whenever a mapping $J \in \mathfrak{A}^{ca}$ exists such that $J^2 = E \in \mathfrak{A}^{cl}$, E trivially belongs to the set $\{L \in \mathfrak{A}^{cl} : L^2 = J^2$ and $LJ = JL\}$, so that this is nonvoid. Thus, (iv) implies (v). Conversely, let $A \in \mathfrak{A}^{ca}$ and let $L \in \mathfrak{A}^{cl}$ be such that $L^2 = A^2$ and $LA = AL$;

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[1] Let us denote by Λ' the centralizer of Λ in K; then we remark that, whenever $\mathfrak{A}$ is a set of additive mappings of X, this statement (i) is equivalent to the statement that $\bar{A}_{\mathfrak{E}} = A$ for any element A of the algebra over $\Lambda \cap \Lambda'$ of mappings of X generated by $\mathfrak{A}$.

[2] We remark that θ coincides with the restriction to $\tilde{G}_2$ of the mapping that will be introduced at the beginning of Proposition 3.

we introduce the mapping $L^{-1}A$ (which obviously belongs to $\mathfrak{A}^{ca}$) and observe that $(L^{-1}A)^2=(L^{-2}A^2)=E$, so that it is involutory. Thus, (v) implies (iv). ∎

PROPOSITION 3. *Let X be a vector space over a division ring K with a conjugation j, and let $\mathfrak{A}$ be a set of mappings of X. With reference to Definitions 1–3, we denote by θ the mapping from $\mathfrak{A}^c$ into the set of the mappings of $\mathfrak{A}^{cl}$ into itself, such that for any $A\in\mathfrak{A}^c$, $\theta(A): L\in\mathfrak{A}^{cl}\to ALA^{-1}\in\mathfrak{A}^{cl}$. Then, the range $\theta(\mathfrak{A}^c)$ of θ is a group of automorphisms of $\mathfrak{A}^{cl}$ which is isomorphic to $\mathfrak{A}^c/(\mathfrak{A}\cup\mathfrak{A}^{cl})^c$, and for any $A\in\mathfrak{A}^{ca}$ we have $\theta(\mathfrak{A}^c)=\theta(\mathfrak{A}^{cl})\cup\theta(\mathfrak{A}^{cl})\circ\theta(A)$ (here and in the sequel $\circ$ denotes composition of mappings of $\mathfrak{A}^{cl}$).*[3]

Moreover, whenever $\mathfrak{A}^{ca}$ is nonvoid, the following conditions are equivalent:[4]

(i) $\theta(\mathfrak{A}^c)=\theta(\mathfrak{A}^{cl})$,

(ii) *for any $A\in\mathfrak{A}^{ca}$, $\theta(A)$ is an inner automorphism of $\mathfrak{A}^{cl}$,*

(iii) *the center $(\mathfrak{A}\cup\mathfrak{A}^c)^c$ of $\mathfrak{A}^c$ (or, equivalently, $(\mathfrak{A}\cup\mathfrak{A}^{cl})^c$) contains antilinear mappings.*

Proof. Since $\mathfrak{A}^c$ is a group and $\theta(A)\circ\theta(B)=\theta(AB)$, θ is an homomorphism of $\mathfrak{A}^c$ onto $\theta(\mathfrak{A}^c)$ whose kernel is $(\mathfrak{A}\cup\mathfrak{A}^{cl})^c$; hence $\theta(\mathfrak{A}^c)$ is isomorphic to $\mathfrak{A}^c/(\mathfrak{A}\cup\mathfrak{A}^{cl})^c$.

Let $\mathfrak{A}^c\neq\mathfrak{A}^{cl}$. We consider the set $\theta(\mathfrak{A}^{ca})$. For any $A,B\in\mathfrak{A}^{ca}$, an element $L\in\mathfrak{A}^{cl}$ exists, because of Proposition 1, such that $B=LA$; hence, $\theta(B)=\theta(L)\circ\theta(A)\in\theta(\mathfrak{A}^{cl})\circ\theta(A)$, that is, $\theta(\mathfrak{A}^{ca})\subseteq\theta(\mathfrak{A}^{cl})\circ\theta(A)$. Conversely, for any $A\in\mathfrak{A}^{ca}$ and $\omega\in\theta(\mathfrak{A}^{cl})\circ\theta(A)$ a mapping $L\in\mathfrak{A}^{cl}$ exists such that $\omega=\theta(LA)\in\theta(\mathfrak{A}^{ca})$; hence, $\theta(\mathfrak{A}^{cl})\circ\theta(A)\subseteq\theta(\mathfrak{A}^{ca})$. Thus, $\theta(\mathfrak{A}^{ca})=\theta(\mathfrak{A}^{cl})\circ\theta(A)$, so that $\theta(\mathfrak{A}^c)=\theta(\mathfrak{A}^{cl})\cup\theta(\mathfrak{A}^{cl})\circ\theta(A)$.

Let us prove now the equivalence of (i), (ii), and (iii). Firstly, let $\theta(\mathfrak{A}^c)=\theta(\mathfrak{A}^{cl})$. Then, for any $A\in\mathfrak{A}^{ca}$, $\theta(A)$ is an inner automorphism of $\mathfrak{A}^{cl}$. Therefore, a linear mapping $S\in\mathfrak{A}^{cl}$ exists such that for any $L\in\mathfrak{A}^{cl}$, we have $ALA^{-1}=SLS^{-1}$; hence, the mapping $B=S^{-1}A\in\mathfrak{A}^{ca}$ also belongs to $(\mathfrak{A}^{cl})^{ca}$, so that $B\in(\mathfrak{A}\cup\mathfrak{A}^{cl})^{ca}$, that is, by using the last statement in Proposition 1, $B\in(\mathfrak{A}\cup\mathfrak{A}^c)^{ca}$. Conversely, let $\theta(\mathfrak{A}^c)\neq\theta(\mathfrak{A}^{cl})$. Then, for any $A\in\mathfrak{A}^{ca}$, $\theta(A)$ cannot be an inner automorphism of $\mathfrak{A}^{cl}$ [in particular $\theta(A)$

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[3]Hence, whenever $\mathfrak{A}^{ca}$ is nonvoid, $\theta(A)$ does not depend on A if and only if $\mathfrak{A}^{cl}$ is commutative.

[4]Observe that $\mathfrak{A}^c$ is commutative if and only if $\theta(\mathfrak{A}^c)=\{1\}$: hence, it is commutative if and only if the above equivalent conditions hold and $\mathfrak{A}^{cl}$ is commutative.

never coincides with 1]. Therefore, the set $(\mathfrak{U}\cup\mathfrak{U}^{cl})^{ca}$ is void [indeed, if not, for any $A\in(\mathfrak{U}\cup\mathfrak{U}^{cl})^{ca}\subseteq\mathfrak{U}^{ca}$ we would have $\theta(A)=1$], so that also $(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}$ is void. Thus, our equivalences are proved together with the equivalences in parentheses in (iii). ■

Proposition 4. *Let X be a vector space over a division ring K with a conjugation j. With reference to Definitions* 1–3, *let $\mathfrak{U}$ be a set of mappings of X; then the following conditions (which imply the equivalent conditions of Propositions* 2 *and* 3)[5] *are equivalent*:

(i) *an involutory antilinear mapping exists which belongs to the center $(\mathfrak{U}\cup\mathfrak{U}^{c})^{c}$ of $\mathfrak{U}^{c}$ (equivalently, to $(\mathfrak{U}\cup\mathfrak{U}^{cl})^{c}$),*[6]

(ii) $(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}=(\mathfrak{U}\cup\mathfrak{U}^{cl})^{ca}$ *is nonvoid, and for any $A\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}$ the set $\{L\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{cl}\colon L^2=A^2\}$ is nonvoid,*

(iii) *$\mathfrak{U}^{c}$ is isomorphic to the direct product of $\mathfrak{U}^{cl}$ and the abstract two-element group G_2.*

Proof. The equivalence in parentheses easily follows by using the statement in parentheses of Proposition 1.

Let us consider statement (i); this coincides with statement (iv) of Proposition 2 with $\mathfrak{U}\cup\mathfrak{U}^{c}$ in the place of $\mathfrak{U}$; hence it is equivalent to statement (v) of Proposition 2 with the same substitution; this last statement, in turn, is equivalent to our statement (ii). Indeed, the latter implies the former, since for any $L\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{cl}$ and $A\in\mathfrak{U}^{c}$, $AL=LA$. Conversely, whenever the former holds [and hence $(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}$ is nonvoid] we can write, recalling Proposition 1, $(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}=(\mathfrak{U}\cup\mathfrak{U}^{c})^{cl}A$, where $A\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}$ is such that the set $\{L\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{cl}\colon L^2=A^2\}$ is nonvoid; hence we easily obtain that for any $B\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}$, a mapping $L\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{cl}$ exists such that $B^2=L^2$, i.e., statement (ii) holds. Thus, we have proved that (i) is equivalent to (ii).

Now let (i) hold; then (iv) of Proposition 2 holds, so that $\mathfrak{U}^{c}=\mathfrak{U}^{cl}\circledS_{\theta}\tilde{G}_2$; moreover, we can set $\tilde{G}_2=\{E,J\}$ with $J\in(\mathfrak{U}\cup\mathfrak{U}^{c})^{ca}$, so that $\theta(J)=1$ and $\mathfrak{U}^{c}=\mathfrak{U}^{cl}\times\tilde{G}_2\sim\mathfrak{U}^{cl}\times G_2$ (here, $\sim$ means group isomorphism). Thus, (i) implies (iii). The converse implication immediately follows by reversing the order of the arguments. ■

[5] Conversely, whenever the equivalent conditions of Propositions 2 and 3 hold, these imply the equivalent conditions listed above if $\mathfrak{U}^{cl}$ is commutative. Indeed, in this case, if $\mathfrak{U}^{c}=\mathfrak{U}^{cl}\circledS_{\theta}G_2$ and θ is inner, the semidirect product necessarily reduces to a direct one.

[6] We note that condition (i) coincides with condition (iv) of Proposition 2 if the set $\mathfrak{U}\cup\mathfrak{U}^{c}$ (equivalently, $\mathfrak{U}\cup\mathfrak{U}^{cl}$) is considered instead of $\mathfrak{U}$; hence, further equivalent conditions arise.

Table 1 summarizes the results obtained in this section. We recall that whenever $\mathfrak{A}^{ca}$ is nonvoid, $\mathfrak{A}^{cl}$ is commutative if and only if for any $A \in \mathfrak{A}^{ca}$, $\theta(A)$ does not depend on A (see footnote 3); if this happens, then also $\mathfrak{A}^{c}$ is commutative in the cases of rows 1 and 4 (see footnote 4), while the case in row 2 never occurs (see footnote 5).

3. THE Λ-LINEAR COMMUTANT OF IRREDUCIBLE SETS OF MAPPINGS

DEFINITION 4 [8, §5, Definition 3]. Let X be a vector space over a division ring K, let $\mathfrak{A}$ be any set of mappings of X, and refer to Definition 3. We denote the set of mappings of X that commute with all the mappings of $\mathfrak{A}$ by $\mathfrak{A}'$, and we call the multiplicative semigroup $\mathfrak{A}'^{l}$ of all the linear mappings of $\mathfrak{A}'$ the *linear commutant* of $\mathfrak{A}$. Furthermore, whenever K is a division ring with a conjugation j, we call the set $\mathfrak{A}'^{\lambda}$ of all the Λ-linear mappings of $\mathfrak{A}'$ the *Λ-linear commutant* of $\mathfrak{A}$, the set $\mathfrak{A}'^{a}$ of all the antilinear mappings of $\mathfrak{A}$ the *antilinear commutant* of $\mathfrak{A}$, and the multiplicative semigroup $\mathfrak{A}'^{h}$ of all the linear and the antilinear mappings of $\mathfrak{A}'$ the *homogeneous commutant* of $\mathfrak{A}$.[7]

DEFINITION 5 [7, Chapter XV, §1, p. 384]. Let X be a vector space over a division ring K. For any set $\mathfrak{A}$ of mappings of X, we say that $\mathfrak{A}$ is *irreducible* if and only if no nonzero $\mathfrak{A}$-invariant proper subspace $Y \subset X$ exists.

PROPOSITION 5. *Let X be a vector space over a division ring K. With reference to Definitions 1–5, let $\mathfrak{A}$ be an irreducible set of mappings which leave the zero element invariant; then the set of nonzero semilinear mappings of X which commute with $\mathfrak{A}$ is a (multiplicative) group. Hence, whenever K is a division ring with a conjugation j, $\mathfrak{A}'^{h} \setminus \{0\} = \mathfrak{A}^{c}$; here 0 is the zero mapping of X.*[8]

Proof. Let $j: \alpha \to \alpha^{j}$ be an automorphism of K, and let B be a nonzero mapping of X which is semilinear with respect to j and commutes with $\mathfrak{A}$.

[7] With reference to Definition 3, note that $\mathfrak{A}^{c} \subset \mathfrak{A}'^{h}$, $\mathfrak{A}^{cl} \subset \mathfrak{A}'^{l}$, $\mathfrak{A}^{ca} \subset \mathfrak{A}'^{a}$.

[8] Whenever $\mathfrak{A}$ is irreducible, Propositions 1, 2, 3, 4 could be restated by using commutants only (in the sense of Definition 4) in the place of centralizers (in the sense of Definition 3) and (in some places) the set $\{0\}$ instead of the empty set.

Let Y be the kernel of B. Then, for any $\alpha, \beta \in K$ and $x, y \in Y$, we have $B(\alpha x+\beta y)=\alpha^j Bx+\beta^j By=0$, that is, $\alpha x+\beta y \in Y$; thus, Y is a vector space over K. Moreover, for any $A \in \mathfrak{A}$, $B(Ax)=ABx=0$, that is, $Ax \in Y$. Thus, Y is invariant under the mappings of $\mathfrak{A}$; then, $\mathfrak{A}$ being irreducible, $Y=\{0\}$, so that B is bijective.

Our statements follow from this.[9] ∎

Definition 6 [8, §3, Definition 2; 11, §11, n^0 2]. Let X be a vector space over a division ring K with a conjugation j. Let $\mathfrak{R}$ be any ring of additive mappings of X. We say that $\mathfrak{R}$ is a *linear-antilinear graded ring* if $\mathfrak{R}=\mathfrak{R}^l+\mathfrak{R}^a$ (the sum is necessarily direct). In this case, the elements of $\mathfrak{R}^h=\mathfrak{R}^l \cup \mathfrak{R}^a$ are called the *homogeneous* elements of $\mathfrak{R}$.

Proposition 6. *Let X be a vector space over a division ring K with a conjugation j. With reference to Definitions 1–6, let $\mathfrak{A}$ be an irreducible set of additive mappings of X; then the following statements are equivalent*:

(i) *an involutory antilinear mapping exists which belongs to $\mathfrak{A}^c$,*
(ii) *$\mathfrak{A}'^l \oplus \mathfrak{A}'^a$ is a linear-antilinear graded ring with divisors of zero.*[10,11]

Proof. For any set $\mathfrak{A}$ of additive mappings of X, $\mathfrak{A}'^l$ is obviously a ring. Moreover, whenever K is a division ring with a conjugation j, $\mathfrak{A}'^a$ is easily verified to be an additive group and $\mathfrak{A}'^l+\mathfrak{A}'^a=\mathfrak{A}'^l \oplus \mathfrak{A}'^a$ to be a linear-antilinear graded ring.

Let (i) hold, i.e., let a $J \in \mathfrak{A}^{ca}$ exist such that $J^2=E$. Then $0 \neq E+J \in \mathfrak{A}'^l \oplus \mathfrak{A}'^a$ and $0 \neq E-J \in \mathfrak{A}'^l \oplus \mathfrak{A}'^a$. Since $(E+J)(E-J)=0$, the mappings $E+J$ and $E-J$ are divisors of zero in the ring $\mathfrak{A}'^l \oplus \mathfrak{A}'^a$. This shows that (i) implies (ii). To prove the converse implication, we assume that $\mathfrak{A}$ is irreducible and put forward some general considerations. More precisely, let $B, C \in \mathfrak{A}'^l \oplus \mathfrak{A}'^a$ and $B \neq 0 \neq C$; we set $B=B_l+B_a$ and $C=C_l+C_a$, where B_l, C_l are linear and B_a, C_a are antilinear mappings of X, and observe that:

[9] We note that the proof may easily be generalized to show that the statement holds even if commutativity is only assumed up to a nonzero factor in K.

[10] We observe that whenever $\mathfrak{A}$ is an irreducible set of additive mappings of a vector space over any division ring K, then $\mathfrak{A}'^l$ is a division ring because of Proposition 5.

[11] Statement (i) coincides with the statement (iv) of Proposition 2. Hence statement (ii) is a further equivalent statement that can be added, whenever $\mathfrak{A}$ is an irreducible set of additive mappings of X, to the ones listed in Proposition 2.

(α) B_l, C_l, B_a, C_a belong to $\mathfrak{A}'^h$; hence either they are zero or they are invertible mappings of X, because of Proposition 5;

(β) since $B\neq 0$, whenever B_l (B_a) is zero, then B_a (B_l) is not zero. The same holds for C.

Now, let (ii) hold and let $BC=0$. Then, we get $BC=(B_lC_l+B_aC_a)+(B_lC_a+B_aC_l)=0$, that is, $B_lC_l+B_aC_a$ being a linear and $B_lC_a+B_aC_l$ an antilinear mapping of X,

$$B_lC_l+B_aC_a=0,$$

$$B_lC_a+B_aC_l=0.$$

Hence, it follows that B_l, C_l, B_a, C_a are not zero. Indeed, if $B_l=0$, we get $B_aC_a=0$, and $B_aC_l=0$. Hence, B_a being nonzero [consideration (β)] and invertible [consideration (α)] we have $C_l=0=C_a$, in contradiction with the assumption $C\neq 0$. Analogously, we can show that B_a, C_l, C_a are not zero. Therefore, we get

$$C_l=-B_l^{-1}B_aC_a,$$

$$B_lC_a-B_aB_l^{-1}B_aC_a=0$$

from the equations written above.

Hence, $B_l=B_aB_l^{-1}B_a$, that is, $B_lB_a^{-1}=B_aB_l^{-1}$. Then let us consider the mapping $J=B_lB_a^{-1}$. This belongs to $\mathfrak{A}^{ca}$; moreover, $J^2=B_lB_a^{-1}B_lB_a^{-1}=B_lB_a^{-1}B_aB_l^{-1}=E$. This shows that (ii) implies (i). ∎

Proposition 7. *Let X be a vector space over a division ring with a conjugation j. With reference to Definitions 1–6, let $\mathfrak{A}$ be a linear-antilinear graded ring of mappings of X;*[12] *then for any $A\in\mathfrak{A}^{ca}$ we have $\mathfrak{A}'^{\lambda}=\mathfrak{A}'^{l}\oplus\mathfrak{A}'^{l}A$.*[13]

Moreover, whenever $\mathfrak{A}$ is irreducible, the subset of the nonzero homogeneous elements of $\mathfrak{A}'^{\lambda}$ is the (multiplicative) group $\mathfrak{A}^{c}$, and $\mathfrak{A}'^{\lambda}$ has divisors of zero if and only if the equivalent statements of Proposition 2 hold.[14]

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[12] It is actually sufficient that $\mathfrak{A}$ be a set of generators of a linear-antilinear graded ring of mappings of X.

[13] In this case the ring $\mathfrak{A}'^{\lambda}$ is obviously an extension, through the element $A\in\mathfrak{A}^{ca}$, of the ring $\mathfrak{A}'^{l}$, and $A\mathfrak{A}'^{l}$ is a set of generators of it (see Proposition 1).

[14] In particular $\mathfrak{A}'^{\lambda}$ is an integral domain, whenever $\mathfrak{A}'^{a}\neq\{0\}$, if and only if: (a) the equivalent statements of Proposition 2 do not hold, (b) $\mathfrak{A}'^{l}$ is commutative, and (c) the equivalent statements of Proposition 3 hold.

Proof. We recall [8, §5, Theorem 4] that the Λ-linear commutant of any linear-antilinear graded ring $\mathfrak{A}$ of mappings of X is $\mathfrak{A}'^{l}\oplus\mathfrak{A}'^{a}$ (hence it is a linear-antilinear graded ring). Whenever $\mathfrak{A}$ is any set of mappings of X such that $\mathfrak{A}^{ca}$ is nonvoid, then, by an obvious extension of the first part of Proposition 1, we obtain $\mathfrak{A}'^{a}=\mathfrak{A}'^{l}A$ (where $A\in\mathfrak{A}^{ca}$). Thus, $\mathfrak{A}'^{\lambda}=\mathfrak{A}'^{l}\oplus \mathfrak{A}'^{l}A$.

Let $\mathfrak{A}$ be irreducible; then, because of Propositions 5 and 6, the remaining parts of our statement easily follow. ■

4. COMMUTATION IN VECTOR SPACES OVER DIVISION RINGS WITH A CONJUGATION OF TYPE II

DEFINITION 7. Let K be a division ring with a conjugation j. We denote the commutant of Λ in K by Λ', and the center of K by K'. We say that K is of *type II* whenever j is inner. (Hence, j is induced by the nonzero elements of a suitable subset $K'i$ of Λ', where $i\in\Lambda'\backslash K'$ and $i^2\in K'$ [8, §4, Theorem 3]).

PROPOSITION 8. *Let X be a vector space over a division ring K with a conjugation j. With reference to Definitions 1–7, let $\mathfrak{A}$ be a set of Λ-linear mappings of X, let K be of type II, and denote an element of $\Lambda'\backslash K'$ which induces j by i. Then $\mathfrak{A}^{ca}=i\mathfrak{A}^{cl}$ [hence $\mathfrak{A}^{ca}$ is nonvoid, the equivalent conditions of Proposition 3 necessarily hold, and conditions (iv) of Proposition 2 and (i) of Proposition 4 can respectively be restated as follows:*

the set $\{L\in\mathfrak{A}^{cl}: i^2L^2=E\}$ is nonvoid,
the set $\{L\in(\mathfrak{A}\cup\mathfrak{A}^{c})^{cl}: i^2L^2=E\}$ is nonvoid[15]].

Proof. Let K be of type II and let $i\in\Lambda'\backslash K'$ induce j as an inner automorphism of K. Then we observe that iE is an antilinear invertible homothety [8, §4, Theorem 3, first observation in the proof] that belongs to $\mathfrak{A}^{ca}$ (since $\Lambda'\subseteq\Lambda$ whenever K is of type II [8, §4, Theorem 3(iii)]) for any set $\mathfrak{A}$ of Λ-linear mappings of X (hence it belongs to $(\mathfrak{A}\cup\mathfrak{A}^{c})^{ca}$).

Making use of this observation, it follows from Proposition 1 that $\mathfrak{A}^{ca}=(iE)\mathfrak{A}^{cl}=i\mathfrak{A}^{cl}$. Hence the statements in parentheses follow easily. ■

Table 2 shows how Table 1 reduces whenever $\mathfrak{A}$ is a set of Λ-linear mappings of X and K is of type II. The symbols in table 2 are used with the

[15] If the assumption that $\mathfrak{A}$ is a linear-antilinear graded ring is added, then it also follows that the Λ-linear commutant $\mathfrak{A}'^{\lambda}$ is the direct sum of $\mathfrak{A}'^{l}$ and $i\mathfrak{A}'^{l}$ (moreover, $\mathfrak{A}'^{\lambda}$ is generated by the additive group $i\mathfrak{A}'^{l}$ coherently with the consideration in footnote 13).

TABLE 2[a]

<table>
<tr>
<td rowspan="2">The equivalent statements of Proposition 2 hold (in particular, a mapping $L \in \mathcal{U}^{cl}$ exists such that $i^2L^2=E$)</td>
<td>The equivalent statements of Proposition 4 hold [in particular, a mapping $L \in (\mathcal{U} \cup \mathcal{U}^c)^{cl}$ exists such that $i^2L^2=E$]</td>
<td>For any $L \in (\mathcal{U} \cup \mathcal{U}^c)^{cl}$ such that $i^2L^2=E$, $\mathcal{U}^c = \mathcal{U}^{cl} \times \{E, iL\}$. (Hence $\mathcal{U}^c$ is commutative if and only if $\mathcal{U}^{cl}$ is also commutative.)</td>
</tr>
<tr>
<td>The equivalent statements of Proposition 4 do not hold [in particular, for any $L \in (\mathcal{U} \cup \mathcal{U}^c)^{cl}$, $i^2L^2 \neq E$]</td>
<td>For any $L \in \mathcal{U}^{cl}$ such that $i^2L^2=E$, $\mathcal{U}^c = \mathcal{U}^{cl} ⓢ_\theta \{E, iL\}$ with $\theta(iL) \neq 1$. (Hence $\mathcal{U}^c$ is noncommutative.)</td>
</tr>
<tr>
<td colspan="2">The equivalent statements of Proposition 2 do not hold (in particular, for any $L \in \mathcal{U}^{cl}$, $i^2L^2 \neq E$)</td>
<td>$\mathcal{U}^c \nsim \mathcal{U}^{cl} ⓢ G_2$. ($\mathcal{U}^c$ is commutative if and only if $\mathcal{U}^{cl}$ is also commutative.)</td>
</tr>
</table>

[a] $\mathcal{U}^{cl} = \mathcal{U}^{cl} \cup i\mathcal{U}^{cl}$.

same meaning as in Table 1 and in Definition 7. (We explicitly note that the 3rd, 5th, and 6th rows of Table 1 do not appear in Table 2 because of the first two statements in parentheses in Proposition 8. Hence, $\mathcal{U}^c$ is commutative if and only if $\mathcal{U}^{cl}$ is also commutative; however, if $\mathcal{U}^{cl}$ is commutative, the case in row 2 never occurs.)

5. EXAMPLES

Firstly, we recall that for any division ring K considered as a left vector space K_s over K itself, the set of linear mappings consists of the set $\underset{\to}{K}$ of the multiplications on the right by the elements of K; furthermore, if K has a conjugation j, the set of antilinear mappings is $\underset{\to}{K} \cdot j$ or, briefly, $\underset{\to}{K}j$ (in the following, we will drop the arrow whenever multiplications on the right and on the left are equivalent).

Let us consider Table 2; all the cases foreseen in it already occur when a one-dimensional vector space over any division ring with a conjugation of type II (for instance, real quaternions) is considered. Indeed, let K be such a ring (it is necessarily noncommutative) and let Λ', K', i be defined as in Definition 7; we assume that the j-invariant subring Λ is a field (hence, $\Lambda' = \Lambda$ [8, §4, Corollary 2]). For any subdivision ring H of K we set

$H_* = H \backslash \{0\}$ and denote by H' the commutant of H in K. Then, if we set $\mathcal{U}_H = \underset{\rightarrow}{H} \cup (i\underset{\rightarrow}{H})j = \underset{\rightarrow}{H} \cup i\underset{\rightarrow}{H}$, the linear-antilinear centralizer of $\mathcal{U}_H$ is $\mathcal{U}_H^c = \underset{\rightarrow}{H}'_* \cup i\underset{\rightarrow}{H}'_*$ [8, §1, p. 133].[16] Thus, three typical cases are (here the symbols $\sim$, Ⓢ, θ, I have the same meaning as in Tables 1 and 2):

(i) $\mathcal{U}_\Lambda = \underset{\rightarrow}{\Lambda} \cup i\underset{\rightarrow}{\Lambda}$,
$\mathcal{U}_\Lambda^c = \underset{\rightarrow}{\Lambda}'_* \cup i\underset{\rightarrow}{\Lambda}'_* = \underset{\rightarrow}{\Lambda}_* \cup i\underset{\rightarrow}{\Lambda}_* = \underset{\rightarrow}{\Lambda}_* \times \{1, i\underset{\rightarrow}{i}^{-1}\}$;
(ii) $\mathcal{U}_{K'} = K' \cup iK'$,
$\mathcal{U}_{K'}^c = \underset{\rightarrow}{K}_* \cup i\underset{\rightarrow}{K}_* = \underset{\rightarrow}{K}_* \circledS_\theta \{1, i\underset{\rightarrow}{i}^{-1}\}$
[$\theta(i\underset{\rightarrow}{i}^{-1})$ is inner and nonidentical];
(iii) $\mathcal{U}_K = \underset{\rightarrow}{K} \cup i\underset{\rightarrow}{K}$,
$\mathcal{U}_K^c = K'_* \cup iK'_* \not\sim K'_* \circledS G_2$.[17]

These examples illustrate the three cases listed in Table 2; hence, they also exemplify the cases listed in rows 1, 2, 4 of Table 1.

We can give a complete exemplification of Table 1 only by considering division rings of type I [8, §4, Theorem 3]. Therefore we take the complex field C with the complex conjugation (which we still denote by j) as the division ring with a conjugation; hence, the j-invariant subring of C is now the real field R.

First, let us consider C as one-dimensional vector space over C itself; then the set of the linear mappings is C and the set of the antilinear mappings is Cj. Now, the following cases occur:

(i) $\mathcal{U}_0 = C$,
$\mathcal{U}_0^c = C_* = \mathcal{U}_0^{cl}$

and

$\mathcal{U}_1 = C \cup Cj$,
$\mathcal{U}_1^c = R_* = \mathcal{U}_1^{cl}$;
(ii) $\mathcal{U}_2 = R$,
$\mathcal{U}_2^c = C_* \cup C_* j = C_* \circledS_\theta \{1, j\}$ [$\theta(j)$ is not inner];
(iii) $\mathcal{U}_\alpha = R \cup R\alpha j$ $(0 \neq \alpha \in C)$,
$$\mathcal{U}_\alpha^c = R_* \cup R_* \alpha j = R_* \times \left\{1, \frac{\alpha}{\sqrt{\alpha\bar{\alpha}}} j\right\}.$$

---

[16] We note that for any set $\mathcal{U}$ of mappings of K_s, $\mathcal{U}^c$ is also the linear-antilinear centralizer of the algebra over $\Lambda \cap \Lambda' = \Lambda$ generated by $\mathcal{U}$; if the mappings of $\mathcal{U}$ are linear or antilinear only, this is a linear-antilinear graded ring which has the form $\underset{\rightarrow}{H} \oplus i\underset{\rightarrow}{H}$, with H a subdivision ring of K and $K' \subseteq H$, and whose homogeneous subset is $\underset{\rightarrow}{H} \cup i\underset{\rightarrow}{H}$. Hence, obviously $\mathcal{U}^c = \underset{\rightarrow}{H}'_* \cup i\underset{\rightarrow}{H}'_*$, so that this expression can be considered as the general form of the centralizer of any linear-antilinear set of mappings of K_s.

[17] $\mathcal{U}_K^c$ cannot be isomorphic to a semidirect product $K'_* \circledS G_2$, since no $\alpha \in K'$ exists such that $\alpha^2 = i^2$, so that no antilinear mapping $A = i\alpha^{-1} \in iK'_*$ exists such that $A^2 = E$ [indeed, should an $\alpha \in K'$ exist such that $\alpha^2 = i^2$, then $\beta = (\alpha + i) \in \Lambda'$ and $\gamma = (\alpha - i) \in \Lambda'$ would be divisors of zero; this is impossible, since Λ' is a field].

These examples respectively illustrate the cases listed in rows 6,3,1 of Table 1.[18]

Secondly, in order to complete our exemplification, we consider a two-dimensional vector space over C and a set $\mathfrak{A}$ of linear mappings which correspond, in a given basis $\mathfrak{S}$, to the set of matrices

$$M(\mathfrak{A})=\left\{\begin{pmatrix} 0 & \alpha \\ -\bar{\alpha} & 0\end{pmatrix}:\alpha\in C\right\}$$

Then we have[19]

$$M(\mathfrak{A}^{cl})=\left\{\alpha\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}:\alpha\in C_*\right\},$$

$$M(\mathfrak{A}^{ca})=\left\{\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}:\alpha\in C_*\right\},$$

hence, $\mathfrak{A}^{c}\not\sim\mathfrak{A}^{cl}\circledS G_2$ and $\theta(A)$ is not inner for any $A\in\mathfrak{A}^{ca}$; thus $\mathfrak{A}^{c}$ now exemplifies the case listed in row 5 of Table 1.

Finally, we notice that all the sets of mappings considered here are irreducible and contain linear and antilinear mappings only, so that the statements of Proposition 7 about the Λ-linear commutant $\mathfrak{A}'^{\lambda}$ may be applied. However, irreducibility is by no means necessary, and the classifications of Tables 1 and 2 apply in more general cases, as concrete examples could easily show. Moreover, we observe that in the examples above $\mathfrak{A}^{cl}$ is sometimes commutative and sometimes not. More detailed exemplifications could be given which show that both situations actually occur in every case foreseen in Table 1 (with the exception of the case in row 2, where $\mathfrak{A}^{cl}$ is necessarily noncommutative).

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REFERENCES

1 R. Ascoli, C. Garola, L. Solombrino, and G. Teppati, Real versus complex representation and linear-antilinear commutant, in *Physical Reality and*

[18]See also Ref. [8], Sec. 1, p. 132. In addition, we observe that for any set $\mathfrak{A}$ of linear and antilinear mappings of C, $\mathfrak{A}^c$ necessarily coincides either with $\mathfrak{A}_0^c$ or with $\mathfrak{A}_1^c$, $\mathfrak{A}_2^c$, $\mathfrak{A}_\alpha^c$; indeed, $\mathfrak{A}^c$ commutes with the algebra over R generated by $\mathfrak{A}$, which is a linear-antilinear graded ring whose homogeneous subset is either $\mathfrak{A}_0$ or $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_\alpha$.

[19]See also Ref. [1], Sec. 4. We recall that for any antilinear mapping A, $M(A)$ is the matrix associated in $\mathfrak{S}$ with the linear mapping $AJ_{\mathfrak{S}}$ (see Definition 2).

Mathematical Description (C. P. Enz and J. Mehra, Eds.), Reidel, Dordrecht, 1974, pp. 239–251.
2 G. Frobenius and I. Schur, *Berl. Berichte*, 1906, pp. 186–208.
3 V. I. Smirnov, *Linear Algebra and Group Theory* (R. A. Silberman, Ed.), McGraw-Hill, New York, 1961.
4 E. P. Wigner, *Group Theory*, Academic, New York, 1959.
5 M. Hamermesh, *Group Theory*, Addison-Wesley, Reading, Mass., 1962.
6 G. Ya. Lyubarskii, *The Application of the Group Theory in Physics*, Pergamon, Oxford, 1960.
7 S. Lang, *Algebra*, Addison-Wesley, New York, 1967.
8 R. Ascoli, C. Garola, L. Solombrino, and G. Teppati, Vector spaces over fields with a conjugation and linear-antilinear commutants, *Rend. Mat. Ser. VI* 10(1):129–148 (1977).
9 F. J. Dyson, The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics, *J. Mathematical Phys.* 3:1199–1215 (1962).
10 R. Ascoli and G. Teppati, On the commutant of irreducible sets of semilinear operators, *Nuovo Cimento B*, LXV(2):229–238 (1970).
11 N. Bourbaki, Algèbre linéaire in *Eléments de Mathématique, Algèbre*, Hermann, Paris, 1967, Chapter 2.

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